

Efficient computation of first passage times in Kou's jump-diffusion model

ABDEL BELKAID¹ and FREDERIC UTZET²

Abstract. S. G. Kou and H. Wang [First Passage times of a Jump Diffusion Process *Ann. Appl. Probab.* **35** (2003) 504–531] give expressions of both the (real) Laplace transform of the distribution of first passage time and the (real) Laplace transform of the joint distribution of the first passage time and the running maxima of a jump-diffusion model called Kou model. Kuo and Wang invert the first Laplace transform by using Gaver-Stehfest algorithm, and the inversion of second one involves a large computing time with an algebra computer system. In the present paper, we give a much simpler expression of the Laplace transform of the joint distribution, and we also show, using Complex Analysis techniques, that both Laplace transform can be extended to the complex plane. Hence, we can use variants of the Fourier-series methods to invert that Laplace transforms, which are very efficient. The improvement in the computing times and accuracy is remarkable.

Keywords: Kou model, Lévy processes, First passage times, Laplace transforms, Bromwich integral.

AMS classification: 60G51, 30D30.

1 Introduction

The Kou model (Kou 2000, and Kou and Wang 2003) is a jump-diffusion model with the jumps given by a mixture of a positive and a negative exponential random variables. It is one of the few Lévy processes, allowing positive and negative jumps, which permits the probability of first passage times to be computed analytically. This property, jointly with the fact that it is very flexible when applied to real data, makes the Kou model very interesting in applications. In Mathematical Finance, in particular, the pricing of important derivative products, as barrier or lookback options, relies on these computations. We should point out that the Kou model belongs to the family of meromorphic Lévy processes (see Kuznetsov 2012 and Kuznetsov *et al.* 2011).

Kou and Wang (2003) obtained a closed expression of the real Laplace transform of the distribution of the first passage time, and an expression of the real Laplace transform of the joint distribution of the process and the running maxima in terms of the so-called Hh functions. They also proposed ways to invert these Laplace transforms. However, for the inversion of the Laplace transform of the distribution of the first passage time they use Gaver-Stehfest algorithm, which has the advantage that it only uses real numbers but, in general, it is difficult to control its accuracy. For the joint distribution, the presence of the Hh functions means that the inversion procedure needs a large computation time with a software package like MATHEMATICA. In the present paper, we first show that the Laplace transform of the first passage time can be extended to the complex plane and can be inverted by using the complex inversion formula or Bromwich integral. Second, we give a closed form of the Laplace transform of the joint distribution of the process and the running maxima without Hh functions, and also prove that it can be inverted by Bromwich integral. As a consequence, very efficient methods can be applied to invert both Laplace transforms. As regards the times reported by Kou and Wang (2003), the computation time of the joint distribution is reduced from minutes to milliseconds.

In Appendix 2 there are the codes of some of the Maple and C programs used by the authors.

¹Department of Production, Technology and Operations Management, IESE Business School, Barcelona Campus, Avda. Pearson, 21, 08034 Barcelona, Spain, e-mail: abelkaid@iese.edu

²Corresponding author. Department of Mathematics, Universitat Autònoma de Barcelona, Campus de Bellaterra, 08193 Bellaterra (Barcelona), Spain, e-mail: utzet@mat.uab.cat

2 Kou model: notations

We use the same notations as Kou and Wang (2003). The Kou model is a Lévy process of the type jump-diffusion of the form

$$X_t = \sigma W_t + \mu t + \sum_{j=1}^{N_t} Y_j, \quad (1)$$

where $\mu \in \mathbb{R}$, $W = \{W_t, t \geq 0\}$ is a standard Brownian motion, $\sigma > 0$, $N = \{N_t, t \geq 0\}$ is a Poisson process of intensity $\lambda > 0$, independent of W , and $\{Y_n, n \geq 1\}$ is a sequence of i.i.d. random variables, independent of W and N , and with a law given by a mixture of a positive and a negative exponential random variables of parameters $\eta_1 > 0$ and $\eta_2 > 0$ respectively; that is, the density of Y_n is

$$f_Y(y) = p\eta_1 e^{-\eta_1 y} \mathbf{1}_{(0, \infty)}(y) + (1 - p)\eta_2 e^{\eta_2 y} \mathbf{1}_{(-\infty, 0)}(y),$$

where $0 < p < 1$. The characteristic function of X_t is

$$\mathbf{E}[e^{isX_t}] = e^{tG(is)}, \quad s \in \mathbb{R},$$

where

$$G(x) = \mu x + \frac{1}{2}\sigma^2 x^2 + \lambda \left(\frac{p\eta_1}{\eta_1 - x} + \frac{(1-p)\eta_2}{\eta_2 + x} - 1 \right). \quad (2)$$

Kou and Wang (2003) Lemma 2.1 prove that for every $\alpha > 0$, the equation $G(x) = \alpha$ has exactly four real zeros: $\beta_{1,\alpha}, \beta_{2,\alpha}, -\beta_{3,\alpha}$ and $-\beta_{4,\alpha}$, where

$$0 < \beta_{1,\alpha} < \eta_1 < \beta_{2,\alpha} \quad \text{and} \quad 0 < \beta_{3,\alpha} < \eta_2 < \beta_{4,\alpha}.$$

See in Figure 1 a typical plot of the function $G(x)$.

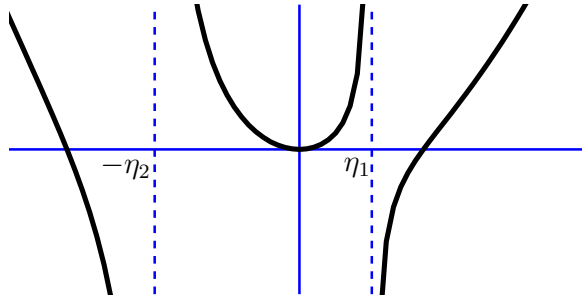


Figure 1. Typical graph of the function $G(x)$

We also have the following equality:

$$\frac{\alpha}{\alpha - G(x)} = \frac{1 - x/\eta_1}{(1 - x/\beta_{1,\alpha})(1 - x/\beta_{2,\alpha})} \frac{1 + x/\eta_2}{(1 + x/\beta_{3,\alpha})(1 + x/\beta_{4,\alpha})}. \quad (3)$$

This equality also holds for $x = is$, $s \in \mathbb{R}$, and indeed it is the Wiener-Hopf factorization related to the Lévy process (see Sato 1999).

3 Theoretical results

3.1 First passage times

For a level $b > 0$ denote by τ_b the first passage time above b :

$$\tau_b = \inf\{t \geq 0 : X_t \geq b\}.$$

We have that $\tau_b \leq t$ if and only if $\max_{0 \leq s \leq t} X_s \geq b$. Our purpose is to compute the probability $\mathbf{P}(\tau_b \leq t)$. To this end, Kou and Wang (2003), Theorem 3.1 and formula on line 16 of page 512, obtained the Laplace transform of that probability (see also Lemma 11 in the Appendix):

Proposition 1. (Kou and Wang 2003) Fix $b > 0$. For every $\alpha > 0$,

$$\int_0^\infty e^{-\alpha t} \mathbf{P}(\tau_b \leq t) dt = \frac{1}{\alpha} \left(\frac{\beta_{2,\alpha}(\eta_1 - \beta_{1,\alpha})}{\eta_1(\beta_{2,\alpha} - \beta_{1,\alpha})} e^{-b\beta_{1,\alpha}} + \frac{\beta_{1,\alpha}(\beta_{2,\alpha} - \eta_1)}{\eta_1(\beta_{2,\alpha} - \beta_{1,\alpha})} e^{-b\beta_{2,\alpha}} \right), \quad (4)$$

where $0 < \beta_{1,\alpha} < \beta_{2,\alpha}$ are the positive zeros of the equation $G(x) = \alpha$.

We prove in the Appendix that this equality can be extended to the half complex plane $\{\alpha \in \mathbb{C} : \text{Re}(\alpha) > 0\}$ except on a finite number of removable singularities. The key point is that by analytic continuation we can construct four analytic functions (of $\alpha \in \mathbb{C}$), that we denote as before by $\beta_{1,\alpha}, \beta_{2,\alpha}, -\beta_{3,\alpha}, -\beta_{4,\alpha}$ which are the four (complex) roots of the equation $G(z) = \alpha$, for $\alpha \in \mathbb{C}$. We prove that for $\alpha \in \mathbb{C}$, with $\text{Re}(\alpha) > 0$, two of these functions always have positive real part and two have negative real part, and we provide a procedure to compute the singular points.

More specifically, $G(z)$ given in (2) is a rational function with numerator of degree 4 and denominator of degree 2, and we can write (see (3))

$$G(z) - \alpha = \frac{P_\alpha(z)}{Q(z)} \quad (5)$$

where $Q(z) = (\eta_1 - z)(\eta_2 + z)$, and $P_\alpha(z)$ is a fourth degree polynomial whose coefficients are polynomials on α (see an example in Section 6), and its roots are the zeros of the equation $G(z) = \alpha$.

In principle, the function defined by the right hand side of (4) has singular points (besides $\alpha = 0$) at the values of α where $\beta_{1,\alpha} = \beta_{2,\alpha}$, that is, where $P_\alpha(z)$ has a multiple root. However, these points are the roots of the resultant $R(\alpha)$ of $P_\alpha(z)$ and $\partial P_\alpha / \partial z$, which is a polynomial in α . The behaviour of the function at these points can be controlled. See the proof of Proposition 2 for details and references.

Proposition 2. For $\alpha \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$, the equality (4) holds, except at some removable singularities given by the roots of the resultant of the polynomials $P_\alpha(z)$ and $\partial P_\alpha / \partial z$, where $P_\alpha(z)$ is given by (5), and $\beta_{1,\alpha}$ and $\beta_{2,\alpha}$ are the zeros of $G(z) = \alpha$ with $\text{Re}(\beta_{1,\alpha}) > 0$ and $\text{Re}(\beta_{2,\alpha}) > 0$.

We prove that the complex Laplace transform (4) can be inverted by using Bromwich integral. Specifically,

Proposition 3. For any $b > 0$ and $t > 0$,

$$\mathbf{P}(\tau_b \leq t) = \frac{1}{2\pi i} \int_{u-i\infty}^{u+i\infty} \frac{1}{\alpha} \left(\frac{\beta_{2,\alpha}(\eta_1 - \beta_{1,\alpha})}{\eta_1(\beta_{2,\alpha} - \beta_{1,\alpha})} e^{-b\beta_{1,\alpha}} + \frac{\beta_{1,\alpha}(\beta_{2,\alpha} - \eta_1)}{\eta_1(\beta_{2,\alpha} - \beta_{1,\alpha})} e^{-b\beta_{2,\alpha}} \right) e^{\alpha t} d\alpha, \quad (6)$$

where $u > 0$ is an arbitrary point such that the vertical line by u does not pass through any singular point mentioned in Proposition 2, and $\beta_{1,\alpha}$ and $\beta_{2,\alpha}$ are the roots of $G(z) = \alpha = u + iv$ with positive real part.

Remark 4. From a practical point of view it is worth remarking that the expression within the integral in (6) is invariant by the permutation of the roots $\beta_{1,\alpha}$ and $\beta_{2,\alpha}$. So, to compute numerically the Bromwich integral in (6) it suffices to compute the four roots of $P_\alpha(z)$ and take the roots with positive real part.

4 Laplace transform of the joint distribution of the process and the running maxima

Our objective is to compute the Laplace transform of the joint probability

$$\mathbf{P}\{X_t \geq a, \tau_b \leq t\} = \mathbf{P}(X_t \geq a, \max_{0 \leq s \leq t} X_s \geq b)$$

for $a \leq b$ and $b > 0$. The variable $\max_{0 \leq s \leq t} X_s$ is called *the running maxima* of the process. Our main result is the following:

Proposition 5. Fix $a \leq b$ and $b > 0$. Then for $\alpha \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$ except at the removable singularities mentioned in Proposition 2,

$$\begin{aligned} \int_0^\infty e^{-\alpha t} \mathbf{P}\{X_t \geq a, \tau_b \leq t\} dt &= \frac{1}{\alpha} (A(\alpha) + B(\alpha)) \\ &\quad + (A(\alpha) C_3(\alpha) + B(\alpha) D_3(\alpha)) e^{-(b-a)\beta_{3,\alpha}} \\ &\quad + (A(\alpha) C_4(\alpha) + B(\alpha) D_4(\alpha)) e^{-(b-a)\beta_{4,\alpha}}, \end{aligned} \quad (7)$$

where

$$A(\alpha) = \mathbf{E}[e^{-\alpha b} \mathbf{1}_{\{X_{\tau_b} = b\}}] = \frac{\eta_1 - \beta_{1,\alpha}}{\beta_{2,\alpha} - \beta_{1,\alpha}} e^{-b\beta_{1,\alpha}} + \frac{\beta_{2,\alpha} - \eta_1}{\beta_{2,\alpha} - \beta_{1,\alpha}} e^{-b\beta_{2,\alpha}}, \quad (8)$$

$$B(\alpha) = \mathbf{E}[e^{-\alpha b} \mathbf{1}_{\{X_{\tau_b} > b\}}] = \frac{(\beta_{2,\alpha} - \eta_1)(\eta_1 - \beta_{1,\alpha})}{\eta_1(\beta_{2,\alpha} - \beta_{1,\alpha})} (e^{-b\beta_{1,\alpha}} - e^{-b\beta_{2,\alpha}}), \quad (9)$$

$$C_j(\alpha) = \frac{1}{\beta_{j,\alpha} G'(-\beta_{j,\alpha})}, \quad j = 3, 4, \quad (10)$$

$$D_j(\alpha) = \frac{\eta_1}{(\eta_1 + \beta_{j,\alpha}) \beta_{j,\alpha} G'(-\beta_{j,\alpha})}, \quad j = 3, 4, \quad (11)$$

$G'(z)$ is the derivative of $G(z)$, and $-\beta_{3,\alpha}$ and $-\beta_{4,\alpha}$ are the zeros of $G(x) = \alpha$ with negative real part.

Remark 6. As in Remark 4, the expression in the right hand side of (7) is invariant by the permutation between the roots with positive real part, and between the roots with negative real part.

Remark 7. Note that with these notations, equality (4) can be written as

$$\int_0^\infty e^{-\alpha t} \mathbf{P}(\tau_b \leq t) dt = \frac{1}{\alpha} (A(\alpha) + B(\alpha)).$$

Also we prove that the Laplace transform (7) can be inverted by Bromwich integral:

Proposition 8. For any $a \leq b$, $b > 0$ and $t > 0$,

$$\begin{aligned} \mathbf{P}\{X_t \geq a, \tau_b \leq t\} = & \frac{1}{2\pi i} \int_{u-i\infty}^{u+i\infty} \left(\frac{1}{\alpha} (A(\alpha) + B(\alpha)) \right. \\ & + (A(\alpha) C_3(\alpha) + B(\alpha) D_3(\alpha)) e^{-(b-a)\beta_{3,\alpha}} \\ & \left. + (A(\alpha) C_4(\alpha) + B(\alpha) D_4(\alpha)) e^{-(b-a)\beta_{4,\alpha}} \right) e^{\alpha t} d\alpha, \end{aligned}$$

where $u > 0$ is an arbitrary point such that the vertical line by u does not pass through any singular point mentioned in Proposition 2, $\beta_{1,\alpha}$ and $\beta_{2,\alpha}$ (respectively $-\beta_{3,\alpha}$ and $-\beta_{4,\alpha}$ are the roots of $G(z) = \alpha = u + iv$ with positive real part (resp., negative part), where G is given in (2), and $A(\alpha)$, $B(\alpha)$, $C_j(\alpha)$ and $D_j(\alpha)$, $j = 3, 4$ are given in (8)-(11).

5 Numerical inversion of the Laplace transforms

5.1 Complex inversion

We follow the classical paper of Abate and Whitt (1995) (see also Abate and Whitt 1992). In order to compute numerically a Bromwich integral of a Laplace transform $\hat{f}(\alpha)$,

$$f(t) = \frac{1}{2\pi i} \int_{u-i\infty}^{u+i\infty} \hat{f}(\alpha) e^{\alpha t} d\alpha,$$

it is first rewritten as a real integral:

$$f(t) = \frac{2e^{ut}}{\pi} \int_0^\infty \operatorname{Re}(\hat{f}(u + ix)) \cos(xt) dx.$$

This integral is numerically evaluated by means of the trapezoidal rule. By taking a step size of length $\pi/(2t)$, half of the cosinus of the sum cancels and we get a nearly alternating series

$$f_d(t) = \frac{e^{A/2}}{2t} \operatorname{Re}(\hat{f})\left(\frac{A}{2t}\right) + \frac{e^{A/2}}{t} \sum_{k=1}^\infty (-1)^k \operatorname{Re}(\hat{f})\left(\frac{A + 2\pi k i}{2t}\right),$$

where $A = 2tu$. Using Poisson summation formula, the discretization error can be controlled. Specifically, when the original function is bounded by 1, $|f(t)| \leq 1$, as in the cases that we are dealing, the discretization error can be bounded:

$$e_d := |f(t) - f_d(t)| \leq \frac{e^{-A}}{1 - e^{-A}} \leq e^{-A}.$$

Then, we can take $A = -\delta \log(10)$ to have at most $10^{-\delta}$ discretization error.

Then the infinite sum is truncated to n terms,

$$s_n(t) = \frac{e^{A/2}}{2t} \operatorname{Re}(\hat{f})\left(\frac{A}{2t}\right) + \frac{e^{A/2}}{t} \sum_{k=1}^n \operatorname{Re}(\hat{f})\left(\frac{A + 2\pi k i}{2t}\right),$$

and finally, as an acceleration method, Euler summation is applied to n terms after an initial B :

$$E(n, B, t) = \sum_{k=0}^n \binom{n}{k} 2^{-n} s_{B+k}(t).$$

To estimate the truncation error, Abate and Whitt (1995) suggest using the difference of successive terms: $E(n, B, t) - E(n+1, B, t)$.

5.2 Inversion of the real Laplace transform by Gaver-Stehfest method

Kou and Wang (2003) invert the real Laplace transform (4) of Proposition 1 by using Gaver-Stehfest method. This method is based on the Post-Widder formula (see Abate and Whitt 1992, page 52) that says that under some regularity conditions

$$f(t) = \lim_{n \rightarrow \infty} f_n(t),$$

where

$$f_n(t) = \frac{(-1)^n}{n!} \left(\frac{n+1}{t} \right)^{n+1} \widehat{f}^{(n)} \left(\frac{n+1}{t} \right),$$

where $\widehat{f}^{(n)}$ is the n -derivative of the Laplace transform \widehat{f} . Gaver changes this derivative by a discrete analog by using finite differences; specifically, he proves (Abate and Whitt 1992, Proposition 8.1) that $f(t) = \lim_{n \rightarrow \infty} \widetilde{f}_n(t)$, where

$$\widetilde{f}_n(t) = \frac{\log 2}{t} \frac{(2n)!}{n!(n-1)!} \sum_{k=0}^n (-1)^k \binom{n}{k} \widehat{f}((n+k) \log(2)/t). \quad (12)$$

To accelerate the convergence, Kou and Wang (2003) use a variation of a method proposed by Stehfest (see Abate and Whitt 1992, Proposition 8.2) computing

$$f_n^*(t) = \sum_{k=1}^n w(k, n) \widetilde{f}_{B+k}(t),$$

where

$$w(k, n) = (-1)^{n-k} \frac{k^n}{k!(n-k)!},$$

and B is the initial burning-out number.

As Abate and Whitt (1992) page 52 remark, Gaver-Stehfest method is much less robust than Fourier-series methods (such as the one based on Bromwich integral), and there is no error analysis. The computation of $f_n^*(t)$ is not easy due to the presence of the factorials in (12). They recommend taking $n = 16$, and using about 30 digits precision.

6 A numeric example

We analyze the example studied by Kou and Wang (2003) with parameters $\mu = 0.1$, $\sigma = 0.2$, $p = 0.5$, $\eta_1 = 50$, $\eta_2 = 100/3$ and $\lambda = 3$. Then $G(z)$ given in (2) is

$$G(z) = 0.1z + 0.02z^2 + 3 \left(\frac{25}{50-z} + \frac{50/3}{100/3+z} - 1 \right).$$

The zeros of $G(z) = \alpha$ are the roots of the polynomial (see 5)

$$P_\alpha(z) = -\frac{1}{50} z^4 + \frac{7}{30} z^3 + (\alpha + 38) z^2 + \left(-\frac{50}{3} \alpha + \frac{425}{3} \right) z - \frac{5000}{3} \alpha,$$

The resultant of P_α and $\partial P_\alpha / \partial z$ is

$$R(\alpha) = -\frac{100}{9} \alpha^5 + \frac{244175}{162} \alpha^4 - \frac{115511483}{1458} \alpha^3 + \frac{3407354201}{1944} \alpha^2 - \frac{17889068323}{972} \alpha - \frac{8877228895}{5832},$$

with zeros, approximately,

$$51.88 + 25.1 i, 15.98 + 15.72 i, -0.08, 15.98 - 15.72 i, 51.88 - 25.1 i.$$

Remark 9. The resultant of two polynomials can be computed easily in most computer algebra systems, for example, in Maple. Anyway, a program in C to compute the resultant can be asked to Oscar Saleta (osr@mat.uab.cat) from the Department of Mathematics of the UAB.

Like Kou and Wang (2003), we consider the values $t = 1$, $b = 0.3$ and $a = 0.2$. The times reported by Kou and Wang (2003) (with a Pentium(R) 400 MHz PC) are given in Table 1. Given that the Laplace transform of the joint distribution provided by Kou and Wang depends on the so-called Hh functions, they need a large computing time with MATHEMATICA.

Table 1. Kou and Wang (2003) results. The accuracy in the computation of $\mathbf{P}(\tau_b \leq t)$ is not provided.

	$\mathbf{P}(\tau_b \leq t)$	$\mathbf{P}(\tau_b \leq t, X_t \geq a)$
Results	0.25584	0.22362
Accuracy	NP	10^{-6}
CPU time	1.76 sec	4.61 min

To do the numeric inversion using Bromwich integral we use C language. With the notations of Subsection 5.1, we take $u = 7$, $A = 14$, $n = 12$ and $B = 4$. The C programs were tested in an Intel(R) Core(TM)2 Duo CPU E8400 @ 3.00GHz with 6GB of ram and Linux OS (Xubuntu 16.10 64bit). The results are given in the first two columns of Table 2. The accuracy is computed comparing the results given in table 2 with the results for $n = 50$.

Table 2. Results using Bromwich integral with $n = 12$ (first two columns) and Gaver-Stehfest with 30 digits precision and $n = 10$ (third column) with a C program.

	$\mathbf{P}(\tau_b \leq t)$	$\mathbf{P}(\tau_b \leq t, X_t \geq a)$	$\mathbf{P}(\tau_b \leq t)$ by Gaver-Stehfest
Results	0.2558436	0.223616	0.2558433
Accuracy	10^{-8}	10^{-7}	
CPU time (ms)	1.2	1.4	33.9

Unfortunately, Kou and Wang (2003) do not provide information about the program or language that they use to compute Gaver-Stehfest algorithm. With the purpose of making a clearer comparison we also program Gaver-Stehfest algorithm with C to estimate the probability $\mathbf{P}(\tau_b \leq t)$ using 30 digits precision, and we get, with $n = 10$, the result given in the third column of Table 2. We should point out that the precision of the Gaver-Stehfest algorithm is more difficult to control since if a bigger n is

Table 3. Results of Gaver-Stehfest algorithm with 30, 40 and 50 digits precision when the number n is increased.

Digits	n					
	10	20	30	40	50	60
30	0.2558433	0.2558430	0.2558430	36238.016	-1.5949e17	-1.0483e31
40	0.2558433	0.2558430	0.2558430	0.2558443	-3.5452e6	-2.2530e20
50	0.2558433	0.2558430	0.2558430	0.2558430	0.2537628	-1.1717e10

used the algorithm starts to oscillate and explodes. The explosion time can be delayed by increasing the number of digits in the computations and in Table 3 there are some results using 30, 40 and 50 digits; this fact is a bit unsatisfactory.

Finally, as another benchmark, we also estimate by Monte Carlo simulation both probabilities $\mathbf{P}(\tau_b \leq t)$ and $\mathbf{P}(\tau_b \leq t, X_t \geq a)$. We should remark that the estimation of probabilities of first passage times of a jump process by Monte Carlo is a difficult problem that would require more research; see the comments by Kou and Wang (2003), pages 520 and 521. For present purposes we generate a discrete approximation of the Kou model along a grid of 2000 points with **R** language, and we make 20000 replications. The results, given in Table 4, are very similar to the ones obtained by Kou and Wang (2003) (last part of their Table 1), and, as they comment, the probabilities are underestimated. The CPU time (with an Intel(R) Core2 Duo 3000 MHz PC) is 7.82 sec.

Table 4. Computation of the probabilities by Monte Carlo simulation with a grid of 2000 points and 20000 replications.

	$\mathbf{P}(\tau_b \leq t)$	$\mathbf{P}(\tau_b \leq t, X_t \geq a)$
Point estimate	0.25195	0.22105
95% IC	(0.2459, 0.2580)	(0.2153, 0.2268)

Remark 10. The codes of Maple programs to compute the probabilities $\mathbf{P}(\tau_b \leq t)$ and $\mathbf{P}(\tau_b \leq t, X_t \geq a)$ by Bromwich integral, and the former also by Gaver-Stehfest algorithm, are given in Appendix 2. Also the C codes to compute both probabilities by Bromwich integral are in that Appendix. The get the C code to compute $\mathbf{P}(\tau_b \leq t)$ by Gaver-Stehfest algorithm with multiprecision, the interested reader can write to Oscar Saleta (osr@mat.uab.cat).

7 Conclusion

We have shown that the Laplace transform of the first passage times of the Kou model can be extended to the complex half plane. We also give an explicit formula of the complex Laplace transform of the joint distribution of the first passage time and the running maxima without the use of the Hh functions. The inversion of this last Laplace transform, compared with the method given by Kou and Wang (2003), produces a dramatic reduction of the computing time. In general the use of complex Laplace transforms instead of real Laplace transforms has the advantage of improving the accuracy and provides a better control of the error.

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Appendix 1: Proof of the propositions

A1. Proof of Propositions 2 and 3

The following property is well known; however, our formulation is not standard and we believe it is convenient to write it out. For the sake of completeness we include a short proof.

Lemma 11. *Let X be a non negative random variable. Then for every $\alpha \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$,*

$$\mathbf{E}[e^{-\alpha X}] = \alpha \int_0^\infty e^{-\alpha x} \mathbf{P}\{X \leq x\} dx. \quad (13)$$

Proof. First we prove the equality for $\alpha > 0$. The proof is based on the property that says that if Z is random variable such that $0 \leq Z \leq 1$, then

$$\mathbf{E}[Z] = \int_0^1 \mathbf{P}\{Z \geq t\} dt,$$

which follows from Fubini Theorem. From that property, for $\alpha > 0$, taking $Z = e^{-\alpha X}$, it is deduced (13).

Now, since both sides of (13) involve Laplace transforms well defined for $\alpha > 0$, both parts can be extended to analytic functions on the half plane $\{\alpha \in \mathbb{C} : \text{Re}(\alpha) > 0\}$; since they coincide for $\alpha > 0$, they coincide on the half plane. ■

Proof of Proposition 2. As we commented after Proposition 1, $P_\alpha(z)$ is a four degree polynomial whose coefficients are polynomials in α . Then, by analytic continuation, we can construct four analytic functions (in α) that give the zeros of this polynomial, see Ahlfors (1979), Chapter 8, Section 2.1. In principle, these analytic functions exist for α in any part of the complex plane where the polynomial $P_\alpha(z)$ has no multiple zeros; since the multiple zeros of $P_\alpha(z)$ are zeros of both $P_\alpha(z)$ and $\partial P_\alpha / \partial z$, it suffices to look for the zeros of the resultant $R(\alpha)$ of $P_\alpha(z)$ and $\partial P_\alpha / \partial z$, which is a polynomial in α . So the four functions are analytic except on the zeros of $R(\alpha)$, but at these points the singularities are removable (see the above mentioned reference to Ahlfors 1979). Hence, removing these singularities we get four analytic functions on the whole \mathbb{C} , that we call the global solutions; in this proof we denote these functions by $\beta_1(\alpha)$, $\beta_2(\alpha)$, $-\beta_3(\alpha)$ and $-\beta_4(\alpha)$, where $\beta_j(\alpha) = \beta_{j,\alpha}$ when $\alpha > 0$ (this happens by the continuity of the global solutions and the fact that for $\alpha > 0$ the four roots of P_α are different, as we mentioned in Section 2).

Now, the expression on the right hand side of (4) can be extended to \mathbb{C} changing $\beta_{j,\alpha}$ by $\beta_j(\alpha)$; that is, define

$$\hat{f}_1(\alpha) = \frac{1}{\alpha} \left(\frac{\beta_2(\alpha)(\eta_1 - \beta_1(\alpha))}{\eta_1(\beta_2(\alpha) - \beta_1(\alpha))} e^{-b\beta_1(\alpha)} + \frac{\beta_1(\alpha)(\eta_1 - \beta_2(\alpha))}{\eta_1(\beta_1(\alpha) - \beta_2(\alpha))} e^{-b\beta_2(\alpha)} \right). \quad (14)$$

This function is analytic except at $\alpha = 0$ and for α such that $\beta_1(\alpha) = \beta_2(\alpha)$. Now we prove that for these last points the singularities are removable. Write \widehat{f}_1 in the following way (we put β_j instead of $\beta_j(\alpha)$ to shorten the notations):

$$\widehat{f}_1 = \frac{e^{\beta_1 b}}{\alpha \eta_1} \left(\beta_1 + \frac{\beta_1(\beta_2 - \eta_1)}{\delta} (e^{-\delta b} - 1) \right),$$

where $\delta = \beta_2 - \beta_1$. Since $e^{-\delta b} - 1 = -\delta b + o(\delta)$, we have that $\lim_{\delta \rightarrow 0} f$ is finite, and so f is bounded in a neighborhood of $\delta = 0$, which implies that the singularity is removable.

Moreover, for $\alpha \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0$, the equation $G(z) = \alpha$ has no roots of the form $z = is$, for some real s ; this is due to the fact that for any $s \in \mathbb{R}$, $\operatorname{Re}(G(is)) < 0$, as can be checked from the expression of $G(z)$ given in (2). This implies that for $\alpha \in \mathbb{C}$, with $\operatorname{Re}(\alpha) > 0$ none of the functions $\beta_1(\alpha)$, $\beta_2(\alpha)$, $-\beta_3(\alpha)$ and $-\beta_4(\alpha)$ crosses the imaginary axis. As a consequence, we always have $\operatorname{Re}(\beta_j(\alpha)) > 0$, $j = 1, \dots, 4$.

On the other hand, since the Laplace transform on the left side of (4) is well defined for $\alpha > 0$, it can be extended to $\alpha \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0$, and it is an analytic function on that half plane. Since this extension and \widehat{f}_1 coincide for $\alpha > 0$ we get the desired result. ■

Proof of Proposition 3. A sufficient condition to apply the complex inversion formula to a function \widehat{f} analytic on \mathbb{C} , except on a finite number of isolated singularities all of them in a half plane $\{\alpha \in \mathbb{C} : \operatorname{Re}(\alpha) \leq a\}$, is that there are positive constants M, R, r such that for $|\alpha| > R$,

$$|\widehat{f}(\alpha)| \leq \frac{M}{|\alpha|^r},$$

see Marsden and Hoffman (1999) Corollary 8.2.2.

We want a bound of this type for the function $\widehat{f}_1(\alpha)$ defined in (14). Extending Kou and Wang (2003), Theorem 3.1, to the complex half plane (or by Lemma 11 and Proposition 1) it follows that for $\alpha \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0$,

$$\mathbf{E}[e^{-\alpha \tau_b}] = \alpha \int_0^\infty e^{-\alpha t} \mathbf{P}(\tau_b \leq t) dt = \alpha \widehat{f}_1(\alpha),$$

Hence,

$$\widehat{f}_1(\alpha) = \frac{1}{\alpha} \mathbf{E}[e^{-\alpha \tau_b}],$$

and since $\operatorname{Re}(\alpha) > 0$ we have $|\mathbf{E}[e^{-\alpha \tau_b}]| \leq 1$. ■

A2. Proof of Propositions 5 and 8

We summarize in the following lemma the computation of the Fourier transform of a rational function that we need:

Lemma 12. *Let R and S be two real polynomials, with $\deg(R) < \deg(S)$, $R(0) \neq 0$, and S with only simple zeros $0, \gamma_1, \dots, \gamma_n$. Then,*

1. *if $\omega < 0$,*

$$\int_{-\infty}^{\infty} e^{-i\omega s} \frac{R(is)}{S(is)} ds = \pi \frac{R(0)}{S'(0)} + 2\pi \sum_{j: \operatorname{Re}(\gamma_j) < 0} \frac{R(\gamma_j)}{S'(\gamma_j)} e^{-\omega \gamma_j}$$

2. if $\omega > 0$,

$$\int_{-\infty}^{\infty} e^{-i\omega s} \frac{R(is)}{S(is)} ds = -\pi \frac{R(0)}{S'(0)} - 2\pi \sum_{j: \operatorname{Re}(\gamma_j) > 0} \frac{R(\gamma_j)}{S'(\gamma_j)} e^{-\omega \gamma_j}$$

Proof.

This is just a particular case of the computation of the Fourier transform of a rational function by calculus of residues. Define $\tilde{R}(z) = R(iz)$ and $\tilde{S}(z) = S(iz)$. Then the rational fraction \tilde{R}/\tilde{S} is analytic except at the poles at $0, -i\gamma_1, \dots, -i\gamma_n$. Applying standard formulas (see, for example, Marsden and Hoffman 1999, Propositions 4.3.11 and 4.3.12) to the computation of $\int_{-\infty}^{\infty} e^{-i\omega s} (\tilde{R}(s)/\tilde{S}(s)) ds$ we get the result. ■

A key point of the proof of Proposition 5 is the following proposition from Kou and Wang (2003):

Proposition 13. (Kou and Wang 2003, Proposition 4.1) Fix $a \leq b$ and $b > 0$. For $\alpha > 0$,

$$\begin{aligned} \int_0^{\infty} e^{-\alpha t} \mathbf{P}\{X_t \geq a, \tau_b \leq t\} dt \\ = A(\alpha) \int_0^{\infty} e^{-\alpha t} \mathbf{P}\{X_t \geq a - b\} dt + B(\alpha) \int_0^{\infty} e^{-\alpha t} \mathbf{P}\{X_t + \xi^+ \geq a - b\} dt, \end{aligned} \quad (15)$$

where ξ^+ is an exponential random variable with parameter η_1 , independent of the process X , and $A(\alpha)$ and $B(\alpha)$ are given in (8) and (9) respectively.

Proof of Proposition 5.

We first compute the Laplace transform for $\alpha > 0$ and later we extend the result to the complex half plane.

A. Computation of the real Laplace transform

1. To compute the first Laplace transform on the right hand of (15), consider, for $c < 0 < d$,

$$\int_0^{\infty} e^{-\alpha t} \mathbf{P}\{c \leq X_t \leq d\} dt = \int_{t=0}^{\infty} \int_{x=c}^d e^{-\alpha t} f(t, x) dx dt, \quad (16)$$

where $f(t, x)$ is the density function of X_t . Denote by ϕ_X the characteristic function of a random variable X . Since the Brownian motion $\{W_t, t \geq 0\}$ and the jumps part of X_t are independent, the characteristic function of X_t satisfies

$$\phi_{X_t}(s) = \mathbf{E}[e^{isX_t}] = e^{tG(is)} = \phi_{\sigma W_t}(s) \tilde{\phi}(s),$$

where $\tilde{\phi}$ is another characteristic function. Then

$$|\phi_{X_t}(s)| \leq |\phi_{\sigma W_t}(s)| = e^{-\sigma^2 s^2 t/2},$$

which is integrable (remember that $\sigma > 0$). Then, by the inversion formula of integrable characteristic functions,

$$f(t, x) = \frac{1}{2\pi} \int_{s=-\infty}^{\infty} e^{-isx} e^{tG(is)} ds.$$

Hence,

$$(16) = \frac{1}{2\pi} \int_{t=0}^{\infty} \int_{x=c}^d \int_{s=-\infty}^{\infty} e^{-\alpha t} e^{-isx} e^{tG(is)} ds dx dt.$$

Furthermore,

$$\begin{aligned}
& \int_{t=0}^{\infty} \int_{x=c}^d \int_{s=-\infty}^{\infty} |e^{-\alpha t} e^{-isx} e^{tG(is)}| dt dx ds \\
&= (d-c) \int_{t=0}^{\infty} e^{-\alpha t} \left(\int_{s=-\infty}^{\infty} e^{-\sigma^2 s^2 t/2} ds \right) dt \\
&= (d-c) \sigma^{-1} \sqrt{2\pi} \int_{t=0}^{\infty} \frac{1}{\sqrt{t}} e^{-\alpha t} dt = (d-c) \sqrt{2\pi} \frac{1}{\sigma \sqrt{\alpha}} < \infty.
\end{aligned}$$

Using Fubini Theorem we can choose the most convenient order to perform the iterated integrals. Therefore,

$$\begin{aligned}
(16) &= \frac{1}{2\pi} \int_{s=-\infty}^{\infty} \left(\int_{t=0}^{\infty} e^{-\alpha t} e^{tG(is)} \left(\int_{x=c}^d e^{-isx} dx \right) dt \right) ds \\
&= \frac{1}{2\pi} \int_{s=-\infty}^{\infty} \frac{1}{is} (e^{-ics} - e^{-ids}) \left(\int_{t=0}^{\infty} e^{-t(\alpha - G(is))} dt \right) ds \\
&= \frac{1}{2\pi} \int_{s=-\infty}^{\infty} \frac{1}{is(\alpha - G(is))} (e^{-ics} - e^{-ids}) ds. \tag{17}
\end{aligned}$$

Now, use the notations in (5):

$$\frac{1}{\alpha - G(is)} = -\frac{Q(is)}{P_{\alpha}(is)},$$

and apply Lemma 12 with

$$R(z) = -Q(z) \quad \text{and} \quad S(z) = zP_{\alpha}(z).$$

Note that from (5),

$$G'(z) = \frac{P'_{\alpha}(z)Q(z) - P_{\alpha}(z)Q'(z)}{Q^2(z)},$$

and hence,

$$G'(\beta_j) = \frac{P'_{\alpha}(\beta_j)}{Q(\beta_j)}, \quad j = 1, 2,$$

and

$$G'(-\beta_j) = \frac{P'_{\alpha}(-\beta_j)}{Q(-\beta_j)}, \quad j = 3, 4. \tag{18}$$

Combining all these formulas, we deduce that

$$(17) = \frac{1}{\alpha} + C_3 e^{c\beta_3} + C_4 e^{c\beta_4} + C_1 e^{-d\beta_1} + C_2 e^{-d\beta_2}, \tag{19}$$

where

$$C_j = C_j(\alpha) = \frac{1}{\beta_{j,\alpha} G'(-\beta_{j,\alpha})}, \quad j = 3, 4,$$

and C_1 and C_2 do not matter.

2. Now we have that

$$\begin{aligned} \int_0^\infty e^{-\alpha t} \mathbf{P}\{X_t \geq a - b\} dt &= \int_0^\infty e^{-\alpha t} \left(\lim_{d \rightarrow \infty} \mathbf{P}\{d \geq X_t \geq a - b\} \right) dt \\ &= \lim_{d \rightarrow \infty} \int_0^\infty e^{-\alpha t} \mathbf{P}\{d \geq X_t \geq a - b\} dt, \end{aligned}$$

where the last equality follows from dominated convergence Theorem. Passing to the limit (19), and taking $c := a - b < 0$, we arrive at expression (7).

The second Laplace transform in (15) is computed exactly in the same way, using that the characteristic function of $X_t + \xi^+$ is

$$\phi_{X_t + \xi^+}(s) = \phi_{X_t}(s) \frac{\eta_1}{\eta_1 - is}.$$

B. Extension to the Laplace transform to the complex half plane. In the same way that in the proof of Proposition 2, the right hand side of (7) can be extended to the complex plane except at some isolated singularities. Denote this function by $\widehat{f}_2(\alpha)$. We have that

$$\widehat{f}_2(\alpha) = \widehat{f}_1(\alpha) + e^{-c\beta_3} \left[\frac{F_3(\alpha)}{\beta_3 G'(-\beta_3)} + \frac{F_4(\alpha)}{\beta_4 G'(-\beta_4)} e^{-c(\beta_4 - \beta_3)} \right],$$

where $\widehat{f}_1(\alpha)$ is the function introduced in the proof of Proposition 2 (see Remark 7), $c = b - a > 0$ and

$$F_j(\alpha) = A(\alpha) + B(\alpha) \frac{\eta_1}{\eta_1 + \beta_j}, \quad j = 3, 4.$$

Given that $\eta_1 > 0$ and $\text{Re}(\beta_j) > 0$, the function \widehat{f}_2 only has singularities where $G'(-\beta_j) = 0$, $j = 3, 4$. However, by (18),

$$G'(-\beta_j) = \frac{P'_\alpha(-\beta_j)}{Q(-\beta_j)}, \quad j = 3, 4,$$

and hence $-\beta_j$ should be a root of both P_α and P'_α , and then $\beta_3 = \beta_4$. As in the proof of Proposition 2, it can be proved that these singularities are removable. ■

Proof of Proposition 8.

We are going to provide a bound of the type given in the proof of Proposition 3 for the function $\widehat{f}_2(z)$ defined on the right hand side of (7):

$$\begin{aligned} \widehat{f}_2(z) &= \frac{1}{\alpha} (A(\alpha) + B(\alpha)) + (A(\alpha) C_3(\alpha) + B(\alpha) D_3(\alpha)) e^{-(b-a)\beta_{3,\alpha}} \\ &\quad + (A(\alpha) C_4(\alpha) + B(\alpha) D_4(\alpha)) e^{-(b-a)\beta_{4,\alpha}}. \end{aligned}$$

Since $A(\alpha) = \mathbf{E}[e^{-\alpha b} \mathbf{1}_{\{\mathbf{x}_{\tau_b} = b\}}]$ and $B(\alpha) = \mathbf{E}[e^{-\alpha b} \mathbf{1}_{\{\mathbf{x}_{\tau_b} > b\}}]$ are bounded by 1, we need only to find bounds for C_j and D_j , $j = 3, 4$. These bounds are deduced from the following estimations of the roots β_3 and β_4 : There is $R > 0$ such that for $|\alpha| > R$, $|\beta_4| > C\alpha^{1/4}$ and $|\beta_3 + \eta_2| < C/|\alpha|$. These estimations are proved below.

From these estimations, it is clear that for $|\alpha| > R$, there is $C > 0$ such that

$$|C_4(\alpha)| = \left| \frac{1}{\beta_4 G'(-\beta_4)} \right| \leq C \frac{1}{|\alpha|^{1/4}}$$

and

$$|D_4(\alpha)| = \left| \frac{\eta_1}{(\eta_1 + \beta_4)\beta_4 G'(-\beta_4)} \right| \leq C \frac{1}{|\alpha|^{1/4}}.$$

In relation to $C_3(\alpha)$ and $D_3(\alpha)$, note that in the expression (5), $Q(z) = (\eta_1 - z)(\eta_2 + z)$, and, see (18),

$$G'(-\beta_3) = \frac{P'_\alpha(-\beta_3)}{Q(-\beta_3)}.$$

Hence, there is $R > 0$ and $C > 0$ such that for $|\alpha| > R$,

$$|C_3(\alpha)| < C/|\alpha| \quad \text{and} \quad |D_3(\alpha)| < C/|\alpha|.$$

The estimations of the roots β_3 and β_4 are deduced from Rouché's Theorem, which says that if f and g are two analytic functions, γ is a closed curve homotopic to a point, and on γ

$$|f(z) - g(z)| < |f(z)|,$$

then f and g have the same number of zeros enclosed by γ (see, for example, Ahlfors 1999, Pag. 153).

To estimate β_3 , note that the polynomial $P_\alpha(z)$ has the form

$$P_\alpha(z) = a_4 z^4 + a_3 z^3 + (a_2 \alpha + a'_2) z^2 + (a_1 \alpha + a'_1) z + a_0 \alpha,$$

where all a_j and a'_j are constants. Then, applying Rouché's Theorem to $f(z) = (a_2 \alpha + a'_2) z^2$, $g(z) = f(z) - P_\alpha(z)$, and γ the circle centered at 0 of radius $C\alpha^{1/4}$, where C is a constant depending on a_j and a'_j , for $|\alpha|$ big enough, we deduce that $P_\alpha(z)$ has two roots inside that circle and two outside. Hence $|\beta_4| > C\alpha^{1/4}$.

To estimate β_3 , apply Rouché's Theorem to $f(z) = \lambda(1-p)\eta_2/(\eta_2 + z) - \alpha$, $g(z) = f(z) - G(z)$ and γ a circle centered at $-\eta_2$ (remember that it is a real number) and radius $C/|\alpha|$ for $|\alpha|$ big enough. ■

Appendix 2. Maple and C codes

Maple Codes

Code 1: Inversion of the Laplace transform of $\mathbf{P}\{\tau_b \leq t\}$ by Bromwich integral: function f1

```

1 Parameters:= proc(mu_, sigma_, lambda_, eta1_, eta2_, p_)
2   global mu, sigma, lambda, eta1, eta2, p;
3   mu := mu_;
4   sigma := sigma_;
5   lambda := lambda_; eta1 := eta1_;
6   eta2 := eta2_;
7   p := p_;
8 end proc;
9
10 Parameters(1/10, 2/10, 3, 50, 100/3, 1/2):
11
```

```

12 Polynomial:= proc()
13   global P;
14   P := (-lambda+mu*z+(1/2)*sigma^2*z^2)*(eta1-z)*(eta2+z)+lambda*(p*eta1*(eta2+z)
15     +(1-p)*eta2*(eta1-z));
16   P := -2*collect(P-(eta1-z)*(eta2+z)*alpha, z)/sigma^2;
17   P := unapply(P, z, alpha);
18   return;
19 end proc:
20
21 Polynomial();
22
23 suma:=proc(f,t,A,n)
24   local sum_a_k,s_n;
25   sum_a_k:=add(evalf((-1)^k*Re(f((A+2*k*Pi*I)/(2*t)))),k=1..n);
26   s_n:=exp(A/2)/(2*t)*Re(f(A/(2*t)))+exp(A/2)/t*sum_a_k;
27   return evalf(s_n);
28 end proc:
29
30 Euler:=proc(f,t,A,n,B)
31   local E,k;
32   E:=2^(-n)*suma(f,t,A,B);
33   for k from 1 to n do
34     E:=E+2^(-n)*binomial(n,k)*suma(f,t,A,B+k);
35   end do;
36 end proc:
37
38 f1:=proc(t,b,A,n,B)
39   local hat_f1;
40   hat_f1:=proc(alpha)
41     local beta,beta1,beta2,denom,num; Polynomial():
42     beta:=fsolve(P(z,alpha)=0,z);
43     beta1:=beta[3];
44     beta2:=beta[4];
45     denom:=alpha*eta1*(beta2-beta1);
46     num:=beta2*(eta1-beta1)*exp(-beta1*b)+beta1*(beta2-eta1)*exp(-beta2*b);
47     return evalf(num/denom);
48   end proc;
49   Euler(hat_f1,t,A,n,B);
50 end proc:
51
52 f1(1,0.3,14,12,4);

```

Lines 1 to 8 fix the parameters of the model. Lines 12 to 19 compute a polynomial with the same roots as P_α . In lines 23 to 28 there is a procedure to compute the discretization of an integral. In lines 30 to 36 the acceleration by Euler summation is computed. Finally, lines 38 to 50 introduce the Laplace transform to be inverted and compute the inversion.

Code 2: Inversion of the Laplace transform of $\mathbf{P}\{X_t \geq a, \tau_b \leq t\}$ by Bromwich integral: function f2

```

1
2 Lines 1 to 36 of Code 1
3
4 f2:=proc(t,a,b,A,n,B)

```

```

5   local hat_f2;
6   hat_f2:=proc(alpha)
7       local G,dG,beta,beta1,beta2,beta3,beta4,aux,Aalpha,Balpha,C3,C4,D3,D4,c;
8       G:=z->mu*z+1/2*sigma^2*z^2+lambda*(p*eta1/(eta1-z)+(1-p)*eta2/(eta2+z)-1);
9       dG:=D(G);
10      beta:=fsolve(P(z,alpha)=0,z);
11      beta3:=-beta[2];
12      beta4:=-beta[1];
13      beta1:=beta[3];
14      beta2:=beta[4];
15      aux:=(beta1,beta2)->(eta1-beta1)*exp(-beta1*b)/(beta2-beta1);
16      Aalpha:=aux(beta1,beta2)+aux(beta2,beta1);
17      Balpha:=aux(beta1,beta2)*(beta2-eta1)/eta1-aux(beta2,beta1)*(eta1-beta1)/eta1;
18      C3:=1/(beta3*dG(-beta3));
19      C4:=1/(beta4*dG(-beta4));
20      D3:=eta1/((eta1+beta3)*beta3*dG(-beta3));
21      D4:=eta1/((eta1+beta4)*beta4*dG(-beta4));
22      c:=a-b;
23      return (Aalpha+Balpha)/alpha+(C3*Aalpha+D3*Balpha)*exp(c*beta3)
24              +(C4*Aalpha+D4*Balpha)*exp(c*beta4);
25  end proc;
26  Euler(hat_f2,t,A,n,B);
27 end proc:
28
29 f2(1,0.2,0.3,14,12,4);

```

Code 3: Inversion of the Laplace transform of $P\{\tau_b \leq t\}$ by Gaver-Stehfest method: function f3

```

1  Digits:=30:
2
3  Lines 1 to 21 of the code 1
4
5  fn:=proc(f,n,t)
6      local suma, k;
7      suma:=0;
8      for k from 0 to n do
9          suma:=suma+ln(2)*(2*n)!/((n-1)!*k!*(n-k)!*t)*(-1)^k*f((n+k)*ln(2)/t);
10     end do;
11     evalf(suma);
12 end proc:
13
14 gaver:=proc(f,t,n,B)
15     local suma, k;
16     suma:=0;
17     for k from 1 to n do
18         suma:=suma+(-1)^(n-k)*k^n/(k!*(n-k)!)*fn(f,k+B,t);
19     end do;
20     evalf(suma);
21 end proc:
22
23
24 f3:=proc(t,b,n,B)
25     local hat_f1;
26     hat_f1:=proc(alpha)

```

```

27     local beta,beta1,beta2,denom,num;
28     beta:=fsolve(P(z,alpha)=0,z);
29     beta1:=beta[3];
30     beta2:=beta[4];
31     denom:=alpha*eta1*(beta2-beta1);
32     num:=beta2*(eta1-beta1)*exp(-beta1*b)+beta1*(beta2-eta1)*exp(-beta2*b);
33     return evalf(num/denom);
34 end proc;
35 gaver(hat_f1,t,n,B);
36 end proc:
37
38 f3(1,0.3,10,2);

```

C Codes

Code 4: Inversion of the Laplace transform of $P\{\tau_b \leq t\}$ by Bromwich integral: function f1

```

1  #include "koujdm_lib.h"
2
3  #include <stdio.h>
4  #include <stdlib.h>
5  #include <inttypes.h>
6
7  int main(int argc, char *argv[]) {
8      Parameters prm;
9      Parametersf1 *prm2 = (Parametersf1 *)malloc(sizeof(Parametersf1));
10
11     if (argc != 12
12         || sscanf(argv[1], "%Lf", &prm.mu) != 1
13         || sscanf(argv[2], "%Lf", &prm.sigma) != 1
14         || sscanf(argv[3], "%Lf", &prm.lambda) != 1
15         || sscanf(argv[4], "%Lf", &prm.eta1) != 1
16         || sscanf(argv[5], "%Lf", &prm.eta2) != 1
17         || sscanf(argv[6], "%Lf", &prm.p) != 1
18         || sscanf(argv[7], "%PRi64, &prm2->t) != 1
19         || sscanf(argv[8], "%Lf", &prm2->b) != 1
20         || sscanf(argv[9], "%PRi64, &prm2->A) != 1
21         || sscanf(argv[10], "%PRi64, &prm2->n) != 1
22         || sscanf(argv[11], "%PRi64, &prm2->B) != 1) {
23         fprintf(stderr, "%s mu sigma lambda eta1 eta2 p t b A n B\n", argv[0]);
24         return -1;
25     }
26
27     long double result = euler(&hat_f1, prm2->t, prm2->A, prm2->n, prm2->B, prm, prm2);
28     printf("result=%30.29Lg\n", result);
29
30     return 0;
31 }

```

Code 5: Inversion of the Laplace transform of $\mathbf{P}\{X_t \geq a, \tau_b \leq t\}$ by Bromwich integral. Function f2

```
1 #include "koujdm_lib.h"
2
3 #include <stdio.h>
4 #include <stdlib.h>
5 #include <inttypes.h>
6
7 int main(int argc, char *argv[]) {
8     Parameters prm;
9     Parametersf2 *prm2 = (Parametersf2 *)malloc(sizeof(Parametersf2));
10
11     if (argc != 13
12         || sscanf(argv[1], "%Lf", &prm.mu) != 1
13         || sscanf(argv[2], "%Lf", &prm.sigma) != 1
14         || sscanf(argv[3], "%Lf", &prm.lambda) != 1
15         || sscanf(argv[4], "%Lf", &prm.eta1) != 1
16         || sscanf(argv[5], "%Lf", &prm.eta2) != 1
17         || sscanf(argv[6], "%Lf", &prm.p) != 1
18         || sscanf(argv[7], "%PRI64", &prm2->t) != 1
19         || sscanf(argv[8], "%Lf", &prm2->a) != 1
20         || sscanf(argv[9], "%Lf", &prm2->b) != 1
21         || sscanf(argv[10], "%PRI64", &prm2->A) != 1
22         || sscanf(argv[11], "%PRI64", &prm2->n) != 1
23         || sscanf(argv[12], "%PRI64", &prm2->B) != 1) {
24         fprintf(stderr, "%s mu sigma lambda eta1 eta2 p t a b A n B\n", argv[0]);
25         return -1;
26     }
27
28     long double result = euler(&hat_f2, prm2->t, prm2->A, prm2->n, prm2->B, prm, prm2);
29     printf("result=%30.29Lg\n", result);
30
31     return 0;
32 }
```

Code 6: Library koujdm_lib.c

```
1 #include "koujdm_lib.h"
2
3 #include <inttypes.h>
4
5 /***
6  * GENERIC FUNCTIONS
7  ***/
8 // binomial coefficient computed recursively
9 int64_t binomial(int64_t n, int64_t k) {
10     if (n < k)
11         return -1;
12     if (k == 0)
13         return 1;
14     if (k > n/2)
15         return binomial(n, n-k);
16     return n*binomial(n-1, k-1)/k;
17 }
```

```

18
19 // coefs. of the polynomial of degree 4
20 Quartic polynomial(long double complex alpha, Parameters prm) {
21     Quartic p;
22     long double sigma2 = prm.sigma*prm.sigma;
23     p.a = 1;
24     p.b = -prm.eta1 + prm.eta2 + 2.*prm.mu/sigma2;
25     p.c = -prm.eta1*prm.eta2 + 2./sigma2*(-prm.mu*prm.eta1 - prm.lambda + prm.mu*prm.eta2 - alpha);
26     p.d = 2./sigma2*(-prm.eta1*prm.eta2*prm.mu - prm.eta1*prm.lambda*(prm.p-1)
27         - prm.eta2*prm.lambda*prm.p + alpha*(prm.eta1-prm.eta2));
28     p.e = 2.*prm.eta1*prm.eta2*alpha/sigma2;
29
30     return p;
31 }
32
33 // direct computation of the roots of the polynomial of degree 4
34 //\protect\vrule width0pt\protect\href{http://math.stackexchange.com/a/786}{http://math.stackexchange.c
35 QuarticSolutions quartic_solve(Quartic p) {
36     long double complex p1 = 2*p.c*p.c*p.c-9*p.b*p.c*p.d+27*p.a*p.d*p.d+27*p.b*p.b*p.e-72*p.a*p.c*p.e;
37     long double complex p2 = p1+csqrt(-4*cpow((p.c*p.c-3*p.b*p.d+12*p.a*p.e),3.)+p1*p1);
38     long double complex p3 = (p.c*p.c-3*p.b*p.d+12*p.a*p.e)/(3*p.a*cpow(p2/2.,1/3.))
39         +cpow(p2/2.,1/3.)/(3*p.a);
40     long double complex p4 = csqrt(p.b*p.b/(4*p.a*p.a)-2*p.c/(3*p.a)+p3);
41     long double complex p5 = p.b*p.b/(2*p.a*p.a)-4*p.c/(3*p.a)-p3;
42     long double complex p6 = (-cpow(p.b/p.a,3.)+4*p.b*p.c/(p.a*p.a)-8*p.d/p.a)/(4*p4);
43     QuarticSolutions result;
44     result.x1 = -p.b/(4*p.a)-p4/2-csqrt(p5-p6)/2;
45     result.x2 = -p.b/(4*p.a)-p4/2+csqrt(p5-p6)/2;
46     result.x3 = -p.b/(4*p.a)+p4/2-csqrt(p5+p6)/2;
47     result.x4 = -p.b/(4*p.a)+p4/2+csqrt(p5+p6)/2;
48     return result;
49 }
50
51 /*
52  * FIRST BLOCK (f1, f2)
53  */
54 // used in f1.c and f2.c
55 long double suma(long double complex (*f)(long double complex, Parameters, void*), int64_t t,
56     int64_t A, int64_t n, Parameters prm, void* prm2) {
57     long double sum_a_k = 0;
58     int64_t sign;
59     for (int_fast64_t k=1;k<=n;k++) {
60         sign = k%2 ? -1 : 1;
61         sum_a_k += sign*creal(f((A+2.*k*M_PI*I)/(2.*t),prm,prm2));
62     }
63     long double s_n = exp(A/2.)/(2.*t)*creal(f(A/(2.*t),prm,prm2))+exp(A/2.)/t*sum_a_k;
64     return s_n;
65 }
66
67 // used in f1.c and f2.c
68 long double euler(long double complex (*f)(long double complex, Parameters, void*), int64_t t,
69     int64_t A, int64_t n, int64_t B, Parameters prm, void* prm2) {
70     long double E = pow(2.,(long double)(-n))*suma(f,t,A,B,prm,prm2);
71     for (int_fast64_t k=1;k<=n;k++) {
72         E += pow(2.,(long double)(-n))*binomial(n,k)*suma(f,t,A,B+k,prm,prm2);
73     }
74     return E;

```

```

75 }
76
77 // used in f1.c
78 long double complex hat_f1(long double complex alpha, Parameters prm, void *prm2) {
79     Parametersf1 *prm2_ = (Parametersf1*)(prm2);
80
81     Quartic p = polynomial(alpha, prm);
82     QuarticSolutions beta = quartic_solve(p);
83
84     long double complex max1,min1,max2,min2;
85
86     if (creal(beta.x1)>creal(beta.x2)) {
87         max1 = beta.x1;
88         min1 = beta.x2;
89     } else {
90         max1 = beta.x2;
91         min1 = beta.x1;
92     }
93     if (creal(beta.x3)>creal(beta.x4)) {
94         max2 = beta.x3;
95         min2 = beta.x4;
96     } else {
97         max2 = beta.x4;
98         min2 = beta.x3;
99     }
100
101     long double complex beta1,beta2;
102     if (creal(max1)<creal(min2)) {
103         beta1 = min2;
104         beta2 = max2;
105     } else if (creal(max2)<creal(min1)) {
106         beta1 = min1;
107         beta2 = max1;
108     } else if (creal(max1)<creal(max2)) {
109         beta1 = max1;
110         beta2 = max2;
111     } else {
112         beta1 = max2;
113         beta2 = max1;
114     }
115
116     long double complex denom = alpha*prm.eta1*(beta2-beta1);
117     long double complex num = beta2*(prm.eta1-beta1)*cexp(-beta1*prm2_->b)
118         +beta1*(beta2-prm.eta1)*cexp(-beta2*prm2_->b);
119     return num/denom;
120 }
121
122 // used in f2.c
123 long double complex aux(long double complex beta1, long double complex beta2, long double eta1,
124     long double b) {
125     return (eta1-beta1)*cexp(-beta1*b)/(beta2-beta1);
126 }
127
128 // used in f2.c
129 long double complex dG(long double complex z, Parameters prm) {
130     return prm.mu+prm.sigma*prm.sigma*z+prm.lambda*(prm.p*prm.eta1/((prm.eta1-z)*(prm.eta1-z))
131         -(1-prm.p)*prm.eta2/((prm.eta2+z)*(prm.eta2+z)));

```

```

132 }
133
134 // used in f2.c
135 long double complex hat_f2(long double complex alpha, Parameters prm, void *prm2) {
136     Parametersf2 *prm2_ = (Parametersf2*)(prm2);
137
138     Quartic p = polynomial(alpha, prm);
139     QuarticSolutions beta = quartic_solve(p);
140
141     long double complex max1,min1,max2,min2;
142
143     if (creal(beta.x1)>creal(beta.x2)) {
144         max1 = beta.x1;
145         min1 = beta.x2;
146     } else {
147         max1 = beta.x2;
148         min1 = beta.x1;
149     }
150     if (creal(beta.x3)>creal(beta.x4)) {
151         max2 = beta.x3;
152         min2 = beta.x4;
153     } else {
154         max2 = beta.x4;
155         min2 = beta.x3;
156     }
157
158     long double complex beta1,beta2,beta3,beta4;
159     if (creal(max1)<creal(min2)) {
160         beta1 = min2;
161         beta2 = max2;
162         beta3 = -max1;
163         beta4 = -min1;
164     } else if (creal(max2)<creal(min1)) {
165         beta1 = min1;
166         beta2 = max1;
167         beta3 = -max2;
168         beta4 = -min2;
169     } else if (creal(max1)<creal(max2)) {
170         beta1 = max1;
171         beta2 = max2;
172         if (creal(-min1)<creal(-min2)) {
173             beta3 = -min2;
174             beta4 = -min1;
175         } else {
176             beta3 = -min1;
177             beta4 = -min2;
178         }
179     } else {
180         beta1 = max2;
181         beta2 = max1;
182         if (creal(-min1)<creal(-min2)) {
183             beta3 = -min2;
184             beta4 = -min1;
185         } else {
186             beta3 = -min1;
187             beta4 = -min2;
188         }
189     }
190 }

```

```

189     }
190
191     complex long double Aalpha, Balpha;
192     Aalpha = aux(beta1,beta2,prm.eta1,prm2_->b)+aux(beta2,beta1,prm.eta1,prm2_->b);
193     Balpha = aux(beta1,beta2,prm.eta1,prm2_->b)*(beta2-prm.eta1)/prm.eta1
194             -aux(beta2,beta1,prm.eta1,prm2_->b)*(prm.eta1-beta1)/prm.eta1;
195
196     long double complex C3,C4,D3,D4;
197     C3 = 1./(beta3*dG(-beta3,prm));
198     C4 = 1./(beta4*dG(-beta4,prm));
199     D3 = prm.eta1/((prm.eta1+beta3)*beta3*dG(-beta3,prm));
200     D4 = prm.eta2/((prm.eta2+beta4)*beta4*dG(-beta4,prm));
201
202     long double c = prm2_->a-prm2_->b;
203
204     return (Aalpha+Balpha)/alpha + (C3*Aalpha+D3*Balpha)*cexp(c*beta3)
205             +(C4*Aalpha+D4*Balpha)*cexp(c*beta4);
206 }

```

Code 7: Library koujdm_lib.h

```

1  #pragma once
2
3  #include <stdio.h>
4  #include <complex.h>
5  #include <math.h>
6  #include <stdarg.h>
7  #include <stdint.h>
8  #include <gmp.h>
9  #include <mpfr.h>
10 #include <mpc.h>
11
12 #define MPCRND MPC_RNDNN
13 #define MPFRND MPFR_RNDN
14
15
16 #ifndef M_PI
17 #define M_PI acos(-1.0)
18 #endif
19
20 /* Structures */
21
22 // quartic polynomial
23 typedef struct {
24     long double complex a,b,c,d,e;
25 } Quartic;
26
27 typedef struct {
28     long double complex x1,x2,x3,x4;
29 } QuarticSolutions;
30
31
32
33 // generic parameter structure
34 typedef struct {

```

```

35     long double mu,sigma,lambda,eta1,eta2,p;
36 } Parameters;
37
38 // specific parameter structures
39 typedef struct {
40     long double b;
41     int64_t t,A,n,B;
42 } Parametersf1;
43
44 typedef struct {
45     long double a,b;
46     int64_t t,A,n,B;
47 } Parametersf2;
48
49
50
51
52 /* Generic functions */
53 // binomial
54 int64_t binomial(int64_t, int64_t);
55 // quartic polynomial
56 Quartic polynomial(long double complex, Parameters);
57 // roots of quartic polynomial
58 QuarticSolutions quartic_solve(Quartic);
59
60
61 // first block functions
62 long double suma(long double complex (*f)(long double complex, Parameters, void*), int64_t, int64_t,
63     int64_t, Parameters, void*);
64 long double euler(long double complex (*f)(long double complex, Parameters, void*), int64_t, int64_t,
65     int64_t, int64_t, Parameters, void*);
66 // f1.c
67 long double complex hat_f1(long double complex, Parameters, void*);
68 // f2.c
69 long double complex aux(long double complex, long double complex, long double, long double);
70 long double complex dG(long double complex, Parameters);
71 long double complex hat_f2(long double complex, Parameters, void*);

```

Code 8: Makefile

```
1 CFLAGS= -std=c11 -Wall -O3
2 LFLAGS= -lm -lgmp -lmpfr -lmpc
3
4 target: all
5
6 all: f1 f2
7
8 # Main executable
9 f1: f1.o koujdm_lib.o
10     gcc $(CFLAGS) f1.o koujdm_lib.o -o f1 $(LFLAGS)
11
12 f2: f2.o koujdm_lib.o
13     gcc $(CFLAGS) f2.o koujdm_lib.o -o f2 $(LFLAGS)
14
15
16
17 # Objects compilation
18 f1.o: f1.c
19     gcc $(CFLAGS) -c f1.c $(LFLAGS)
20
21 f2.o: f2.c
22     gcc $(CFLAGS) -c f2.c $(LFLAGS)
23
24
25 # Run configuration
26 runf1: f1
27     ./f1 0.1 0.2 3 50 33.33333333333333 0.5 1 0.3 14 12 4
28
29 runf2: f2
30     ./f2 0.1 0.2 3 50 33.33333333333333 0.5 1 0.2 0.3 14 12 4
31
32
33 # Cleaning directives
34 clean:
35     rm *o f1 f2
```

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